Province de Liège
Enseignement

HAUTE ECOLE DE LA PROVINCE DE LIEGE

A characterization of the lattice of convex bodies

Fourneau René

Gruber's characterisation problem

In the *Handbook of Convex Geometry*, P. M. Gruber cites some of my papers related to the lattice of all closed bounded convex subsets of a locally convex Hausdorff topological vector space and asks for a characterization of this lattice .

304 *P.M. Gruber*

Fourneau (1984a) gave a characterization of those pairs $C, D \in \mathcal{K}$ for which C is *subperspective* to D, i.e.,

$$C \subset D \vee E, \quad C \wedge E = \emptyset \qquad \text{for suitable } E \in \mathcal{K}.$$

Filters were considered by Fourneau (1984b), and Fourneau (1976a) also investigates relations between lattice isomorphisms and semigroup isomorphisms from $\mathcal{K}(\mathbb{E}^c)$ onto $\mathcal{K}(\mathbb{E}^d)$. (For the semigroup structure see below.)

It seems that many further results on lattices of convex bodies still wait for their discovery. One problem is the following: characterize the lattice $\langle \mathcal{K}, \wedge, \vee \rangle$ or sublattices of it by algebraic and geometric properties.

Let us remind that, in a locally convex Hausdorff topological vector space E, the set $\mathcal{B}(E)$ of all closed bounded convex subsets of E is endowed with a lattice structure by the operations :

$$A \wedge B = A \cap B$$
$$A \vee B = \mathcal{C}(A \cup B)$$

where $\mathcal{C}()$ denotes the closed convex hull.

Uniqueness

Concerning the uniqueness of the representation of a lattice as the lattice $\mathcal{B}(E)$ for some locally convex Hausdorff topological vector space E, a previous theorem of mine settles the question :

For locally convex Hausdorff topological vector spaces E_1 and E_2, the lattices $\mathcal{B}(E_1)$ and $\mathcal{B}(E_2)$ are isomorphic if and only if E_1 and E_2 are isomorphic in their weak topologies :

$$\mathcal{B}(E_1) \xrightarrow{\sim} \mathcal{B}(E_2)$$
$$\Updownarrow$$
$$(E_1, \sigma(E_1, E_1^*)) \xrightarrow{\sim} (E_2, \sigma(E_2, E_2^*))$$

Existence : the construction

Let us now turn to the existence part of the characterization problem in finite dimension.

Assume that $\mathcal{L}$ is the given lattice. We can now turn to a paper of M. K. Bennett, where she characterizes the lattice of all convex subsets of a vector space on $\mathbb{R}$.

Bennett assumes that $\mathcal{L}$ is a complete atomic $\vee$-continuous lattice with joint dense atoms satisfying eight axioms involving only atoms (axioms 1 to 8 hereafter). She then lets E be the set of all atoms of $\mathcal{L}$ and defines segments

$$[a : b] = \{c \in E \,:\, c \leq a \vee b\}$$

and straight lines

$$(a : b) = \{c \in E \,:\, c \leq a \vee b, \quad a \leq b \vee c, \quad b \leq a \vee c\}.$$

We modify this construction by releasing the completeness and $\vee$-continuity hypothesises : they are just needed to insure that the supremum of a straight line is an element of $\mathcal{L}$, a property that we do not need. A vectorialization of E on the real field is obtained this way, the linear segments of which coincide with the segments previously defined. The elements of $\mathcal{L}$, identified with the set of all atoms they are made of, are thence convex subsets of E for this vectorialization.

We now have to translate the *closed* and *bounded* properties in lattice terms.

For this, we use two properties of finite-dimensional real vector spaces :

- *a convex subset A is closed if and only if* $[a : b[\subset A$ *implies* $[a : b] \subset A$ *(A is* algebraically closed) ;
- *a subset is bounded if and only if it is included in a convex polytope.*

One can also limit the dimension of E by imposing, in terms of atoms and lattice operations, that no simplex can have more than a fixed number of vertices.

Before to state the characterization theorem, let us remind two definitons due to Bennett :

- $a \vee b$ and $c \vee d$ are said to be *skew* if $\forall e,\ f,\ g,\ h$ such that $a \vee b \leq e \vee f$ and $c \vee d \leq g \vee h$, then $(e \vee f) \wedge (g \vee h) = 0$;
- $a \vee b$ and $c \vee d$ are said to be *parallel* if and only if $a \vee b$ and $c \vee d$ are skew and $$[(a \vee c) \wedge (b \vee d)] \vee [(a \vee d) \wedge (b \vee c)] \neq 0.$$

We shall write $(a \vee b) \sim (c \vee d)$ to note that $a \vee b$ and $c \vee d$ are skew and $(a \vee b) \parallel (c \vee d)$ to note that they are parallel.

The Characterization theorem

Theorem. *Let $\mathcal{L}$ be an atomic and atomistic lattice satisfying the following axioms (lower case letters stand for atoms and $\overline{P}$ is the negation of P) :*

1. $[c, d \leq a \vee b,\ c \neq d] \Rightarrow \left[c \leq a \vee d \Leftrightarrow \overline{c \leq b \vee d}\right]$;
2. $[c \leq a \vee e,\ d \leq b \vee c] \Rightarrow [(b \vee c) \wedge (a \vee d) \neq 0]$;
3. $[a \leq x_1 \vee \ldots \vee x_r] \Rightarrow [\exists b \leq x_1 \vee \ldots \vee x_{r-1}, \quad a \leq b \vee x_r]$;
4. $\left[\forall a, b,\ \exists c, \quad b \leq a \vee c \quad \& \quad \overline{c \leq a \vee b}\right]$;
5. $\exists a, b, c$ *and* d *such that* $a \vee b$ *and* $c \vee d$ *are skew and non parallel* ;
6. $[c \wedge (a \vee b) = 0,\ a \wedge (b \vee c) = 0 \quad \& \quad b \wedge (a \vee c) = 0] \Rightarrow [\exists d$ *such that* $(c \vee d) \parallel (a \vee b)$; *and if* $(c \vee e) \parallel (a \vee b)$, $\exists f, g$ *such that* $(c \vee d \vee e) \leq (f \vee g)]$;
7. *if* $y \leq x \vee z$, $\exists y_0 = x,\ y_1, \ldots, y_r = y$ *such that* $\left(y_i \vee y_{i+1}\right) \sim \left(y_{i+1} \vee y_{i+2}\right)$, $i = 0, \ldots, r-2$ *et* $z \leq y_{r-1} \vee y_r$;
8. *if* $\{x_\alpha, y_\alpha\}_{\alpha \in A}$ *is a set of atoms such that* $a \leq x \vee y \Rightarrow \exists \alpha \in A$ *such that* $a \leq x_\alpha \vee y_\alpha$, $a \neq x_\alpha, y_\alpha$; *if, moreover,* $\forall d, e, f \in \{x_\alpha, y_\alpha\}$, $\exists g, h$ *such that* $d \vee e \vee f \leq g \vee h$, *then* $\exists B \subset A$, B *finite, such that* $a \leq x \vee y \Rightarrow \exists \alpha \in B$ *such that* $a \leq x_\alpha \vee y_\alpha$, $a \neq x_\alpha, y_\alpha$;
9. $\exists n \in \mathbb{N}_0$ *such that in every set* A *of more than* $n+2$ *atoms one can find* a *and* b *for which* $$(a \vee b) \wedge \left[\bigvee (A \setminus \{a, b\})\right] \neq 0 \,;$$
10. *if* $B \in \mathcal{L}$ *follows every* $c \leq a \vee b$, $c \neq b$, *then* $b \leq B$;
11. *every element of $\mathcal{L}$ is followed by the supremum of a finite family of atoms.*

The lattice $\mathcal{L}$ is isomorphic to $\mathcal{B}(E)$, where E is a n-dimensional real vector space.

References

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